Discrete Mathematics 207 (1999) 53–63

DISCRETE MATHEMATICS

On the existence of magic n-dimensional rectangles

Thomas R. Hagedorn

Department of Mathematics and Statistics, The College of New Jersey, P.O. Box 7718, Ewing, NJ 08628-0718, USA

Received 6 May 1998; revised 16 December 1998; accepted 4 January 1999

Abstract

Magic rectangles are a classical generalization of the well-known magic squares. In this paper, we generalize magic rectangles to n dimensions. We demonstrate necessary conditions for magic n-rectangles to exist and in a large number of cases, we show that these conditions are sufficient.

MSC: 05B15

Keywords: Magic squares; Magic rectangles

1. Introduction

Magic rectangles are a natural generalization of the magic squares which have long intrigued mathematicians and the general public. A $m \times n$ magic rectangle is a $m \times n$ array in which the first mn positive integers are placed so that the sum of the entries of each row is constant and each column sum is another (different if $m \neq n$) constant. Harmuth studied magic rectangles over a century ago and proved.

Theorem 1 (Harmuth [7,8]). *For m, $n > 1$, there is a $m \times n$ magic rectangle if and only if $m \equiv n \bmod 2$ and $(m,n) \neq (2,2)$.*

Recently, Sun [9], Bier and Rogers [3], Bier and Kleinschmidt [4], and the author [6] have published simplified modern proofs of Harmuth's result.

In this paper, we study the generalization of magic rectangles to n dimensions. Suppose $R = (r_i)$ is a n-dimensional $m_1 \times \cdots \times m_n$ array whose entries are the first $m_1 \dots m_n$ positive integers. We index the entries r_i of R by the multi-index $i=(i_1,\dots,i_n)$. We call R a magic n-rectangle of size $(m_1,\dots,m_n)$ if for each k, the row sum over the kth index $\sum_{i_k=1}^{m_k} r_i$ is independent of the choice of the indices i_j for $j \neq k$. A magic

E-mail address: hagedorn@math.tcnj.edu (T.R. Hagedorn)

0012-365X/99/$ - see front matter
PII: S0012-365X(99)00040-0

n-cube of size m is a magic n-rectangle whose lengths m_i are all equal to m and where one imposes the additional condition that the sums along the longest diagonals must equal the sums of the rows.

Recently, magic n-cubes have been studied by several authors. For m, $n > 2$, Sun showed in [10] that magic n-cubes of size m always exist. Furthermore, Adler [1] found a surprising connection between the 3-adic L-functions of number theory and magic cubes. Adler and Washington further extended this connection in [2] to construct magic p-cubes using p-adic L-functions. In both of these constructions, the natural symmetries of the n-cube are used. Consequently, these methods will not construct magic n-rectangles.

In this paper, we show the existence of magic n-rectangles in some special cases. In Section 2, we show that a magic n-rectangle can only exist if its lengths m_i all have the same parity. We can thus divide the set of magic n-rectangles into even and odd magic n-rectangles. In Section 3, we prove

Theorem 2. *If m_i are positive even integers with $(m_i,m_j)\neq(2,2)$ for $i\neq j$, then a magic n-rectangle of size $(m_1,\dots,m_n)$ exists.*

We prove Theorem 2 by using centrally symmetric rectangles, a class of rectangles closely related to magic rectangles. We study them in Section 2. Centrally symmetric rectangles have the property that they can be added together or to magic rectangles to form larger rectangles of the same type. Theorem 2 can then be proved using induction.

As is the case when $n=2$ (see [6]), odd magic n-rectangles are harder to construct than the even magic n-rectangles. In Section 4, we construct a subset of the set of odd magic 3-rectangles:

Theorem 3. *Let m_1, m_2, $m_3>1$ be three odd numbers with $gcd(m_1,m_2)>1$. Then there exists a magic 3-rectangle of size (m_1,m_2,m_3).*

As a result, magic 3-rectangles of sizes $(3,3,3)$, $(3,3,5)$ and $(3,5,15)$ do exist. However, it remains an open question whether a magic 3-rectangle of size $(3,5,7)$ exists.

2. General properties

In this section, we will prove some general statements about magic n-rectangles. We will also introduce the notion of a centrally symmetric rectangle and prove results that will allow us to use them to construct magic n-rectangles. These results are essential to proving the existence of even magic n-rectangles in Theorem 2.

Proposition 1. *If there exists a magic n-rectangle with size $(m_1,\dots,m_n)$ then $m_i\equiv m_j \bmod 2$ for all i, j.*

Proof. Suppose that $R=(r_i)$, $i=(i_1,\dots,i_n)$ is the given magic rectangle. If we let $N=m_1\dots m_n+1$, then the sum of the first $m_1\dots m_n$ positive integers is $N(N-1)/2$ and the row sum for the kth index i_k of R is $Nm_k/2$. Now suppose one of the lengths m_i is even. Then N is odd and since the sums $Nm_k/2$ are integers, all the lengths m_k must be even.

Remark. In the case $n=2$, one traditionally refers to the row and column sums of a rectangle. However, n-dimensional rectangles require us to refer to the sums as row sums. The column and row sums of a 2-rectangle would then be the row sums over the first and second indices, respectively.

We can now divide up the set of magic n-rectangles into the sets of odd and even rectangles. We shall see that some even magic n-rectangles do not exist.

Proposition 2. *For* $n>1$, *there does not exist a magic n-rectangle of size* $(2,2,m_3,\dots,m_n)$.

Proof. Suppose $R=(r_i)$ is magic n-rectangle of size $(2,2,m_3,\dots,m_n)$. Then the sums $r_{11i'}+r_{21i'}$ and $r_{11i'}+r_{12i'}$ must be equal and so $r_{21i'}=r_{12i'}$. But the entries in a magic rectangle are distinct and thus R cannot exist.

We will see in Theorem 2 that these are the only even magic n-rectangles which do not exist. All the other possible even magic n-rectangles can be constructed.

We now turn our attention to constructing magic n-rectangles. We will see that one can use magic n-rectangles to build magic rectangles of larger sizes and higher dimensions. Generalizing [3, Theorem 1], we can multiply magic n-rectangles together.

Theorem 4. *If there are magic n-rectangles of sizes* $(m_{k,1},\dots,m_{k,n})$ *for* $k=1,\ 2$, *then there is a magic n-rectangle of size* $(m_{1,1}m_{2,1},\dots,m_{1,n}m_{2,n})$.

Proof. We follow the proof of [3, Theorem 1]. Let $S=(s_j)$ and $T=(t_l)$ be the two given magic n-rectangles. We define an array $R=(r_i)$ of size $(m_{1,1}m_{2,1},\dots,m_{1,n}m_{2,n})$ by

$$r_i=s_j+(t_l-1)m_{1,1}\cdots m_{1,n}$$

where the kth index i_k is given by the formula $i_k=j_k+(l_k-1)m_{1,k}$. Here j_k ranges from 1 to $m_{1,k}$ and l_k ranges from 1 to $m_{2,k}$. It is easy to check that the row sums of R are constant and that the entries of R are the first positive $m_{1,1}m_{2,1}\cdots m_{1,n}m_{2,n}$ integers. Hence, R is a magic n-rectangle.

We now assume that the positive integers m_i are even. To show the existence of magic n-rectangles, it is convenient to introduce centrally symmetric rectangles. Our definition is a variant of that introduced in [4].

Definition 1. For $x > -1$, let R be an n-dimensional $m_1 \times \cdots \times m_n$ rectangular array whose entries are the numbers $\pm(x+1), \ldots, \pm(x+M/2)$, where $M = m_1 \cdots m_n$. We call R a centrally symmetric n-rectangle of size $(m_1, \ldots, m_n)$ and type x if the sum of all the rows is zero. Additionally, we say that R is balanced if R has an equal number of positive and negative numbers in each row.

If R is an even magic n-rectangle of size $(m_1, \ldots, m_n)$, then by subtracting $(M+1)/2$ from each entry of R, we obtain a centrally symmetric n-rectangle of type $-1/2$. Similarly, every centrally symmetric n-rectangle of type $-1/2$ gives rise to an even magic n-rectangle with the same size. Thus, we can use the existence of centrally symmetric n-rectangles to prove results about even magic n-rectangles.

Proposition 3. *If there exists a balanced centrally symmetric n-rectangle, then there exists a magic n-rectangle of the same size.*

In order to prove Proposition 3, we will need the following lemma. Lemma 1 shows that the existence of a balanced centrally symmetric n-rectangle does not depend on the choice of its type. Consequently, in subsequent sections, when we discuss existence questions for balanced symmetric n-rectangles, we will not refer to the type.

Lemma 1. *For $x,\ y > -1$, if there exists a balanced centrally symmetric n-rectangle of type x, then there exists a balanced centrally symmetric n-rectangle of the same size with type y.*

Proof. Suppose $R = (r_i)$ is the given rectangle and let $M = m_1 \cdots m_n$. Then define a n-rectangle $S = (s_i)$ of the same size by $s_i = (y-x)\mathrm{sgn}(r_i) + r_i$. The entries of S are the numbers $\pm(y+1), \ldots, \pm(y+M/2)$. Now the row sum for the kth index is

$$\sum_{i_k} s_i = (y-x)\sum_{i_k} \mathrm{sgn}(r_i) + \sum_{i_k} r_i = 0.$$

Hence S is a centrally symmetric n-rectangle of type y. If r_i is positive, then $r_i = x + j$ for some $j \geqslant 1$. Hence $s_i = y + j$ is also positive. Similarly, r_i negative implies s_i negative. Hence S is balanced.

Proof of Proposition. Suppose R is the given rectangle. If R has type x, then by Lemma 1, there exists a balanced centrally symmetric n-rectangle with the same size and with type $-1/2$. Hence there exists a magic n-rectangle with the same size. □

Centrally symmetric rectangles are more useful to work with than magic rectangles because in certain situations, they can be added together.

Proposition 4. *If there exists a balanced centrally symmetric n-rectangle R of size $(a, m_2, \ldots, m_n)$ and a centrally symmetric n-rectangle S of size $(b, m_2, \ldots, m_n)$ and type*

x, then there exists a centrally symmetric n-rectangle T of size $(a+b, m_2, \dots, m_n)$ and type x. If S is a balanced rectangle, then T can be chosen to be balanced as well.

Proof. By Lemma 1, we know there exists a balanced centrally symmetric rectangle R' with the same size as R of type $x + (bm_2 \cdots m_n)/2$. Then by stacking R' and S together, we obtain a rectangle T whose rows sum to zero. Hence T is a centrally symmetric rectangle with size $(a+b, m_2, \dots, m_n)$, and type x. If S is balanced, it is easy to see that T is as well.

Since magic n-rectangles correspond to centrally symmetric rectangles of type $-1/2$, we have the following corollary.

Corollary 1. *Suppose there exists a magic n-rectangle with size $(a, m_2, \dots, m_n)$ and a balanced centrally symmetric rectangle with size $(b, m_2, \dots, m_n)$, then there exists a magic n-rectangle of size $(a+b, m_2, \dots, m_n)$.*

We can also use centrally symmetric n-rectangles to create centrally symmetric rectangles and magic rectangles of higher dimensions.

Proposition 5. *If there exists a centrally symmetric n-rectangle of size $(m_1, \dots, m_n)$ and type $-1/2$, then there exists a balanced centrally symmetric $n+1$-rectangle of size $(4, m_1, \dots, m_n)$.*

Proof. Let $R = (r_1)$, with $i = (i_1, \dots, i_n)$ be a centrally symmetric n-rectangle of type $-1/2$. Let $M = m_1 \cdots m_n$. We define $S = (s_j)$ by

$$s_{1i} = \begin{cases} r_i + M/2 & \text{if } \sum_k i_k \text{ is even,} \\ r_i - M/2 & \text{otherwise.} \end{cases}$$

$$s_{2i} = \begin{cases} -r_i + 3M/2 & \text{if } \sum_k i_k \text{ is even,} \\ -r_i - 3M/2 & \text{otherwise,} \end{cases}$$

and $s_{3i} = -2r_i - s_{2i}$, $s_{4i} = 2r_i - s_{1i}$. The entries of S are $\pm 1/2, \dots, \pm(2M - 1/2)$. Since the dimensions of R are even, for any constants a, k, $\sum_{i_k} s_{ai} = \sum_{i_k} r_i = 0$. Also, $\sum_a s_{ai} = 0$ and it is easy to check that each row of S contains an equal number of odd and even numbers. Hence S is a balanced centrally symmetric rectangle of type $-1/2$.

Using Lemma 1, we have

Corollary 2. *If there exists a balanced centrally symmetric n-rectangle of size $(m_1, \dots, m_n)$, then there exists a balanced centrally symmetric $n+1$-rectangle of size $(4, m_1, \dots, m_n)$.*

3. Even magic n-rectangles

We now consider the question of whether an even magic n-rectangle of size $(m_1,\dots,m_n)$ exists. We will use certain 'small' centrally symmetric rectangles as building blocks to construct the even magic n-rectangles.

Proposition 6. *For $n>1$, there exists a balanced centrally symmetric n-rectangle of size $(4,\dots,4,2)$.*

Proof. The 2×4 array

$$A=\begin{pmatrix} 1 & -2 & -3 & 4 \\ -1 & 2 & 3 & -4 \end{pmatrix}$$

is a balanced centrally symmetric 2-rectangle of size $(2,4)$. The proposition then follows by using induction and Corollary 2.

We now prove

Theorem 5. *Let m_i be positive even integers with $(m_i,m_j)\neq(2,2)$ for $i\neq j$. If a magic m-rectangle of size $(2,6,\dots,6)$ exists for all $1<m\leqslant n$, then a magic n-rectangle of size $(m_1,\dots,m_n)$ exists.*

Proof. When $n=2$, the existence of magic rectangles was established in [3,6,7,9]. We now induct on n and assume the theorem is established for $n-1$. Since a balanced centrally symmetric n-rectangle of size $(2,4,\dots,4)$ exists by Proposition 6, a magic n-rectangle of the same size also exists by Proposition 3. Let $m=(m_1,\dots,m_n)$ be a valid size for which the existence of a magic n-rectangle of size m has not been established. We can assume that the size is chosen so that $\sum m_i$ is minimal. Suppose that $m_1=4$. Now by induction we know that a magic $n-1$-rectangle of size $(m_2,\dots,m_n)$ exists. Proposition 5 then shows that a balanced centrally symmetric n-rectangle of size $(m_1,\dots,m_n)$ exists. Thus a magic n-rectangle of the same size exists. Hence we can assume that $m_i\neq 4$ for all i. Now suppose that $m_1>6$. We have just shown that a balanced centrally symmetric n-rectangle of size $(4,m_2,\dots,m_n)$ exists. Also, since the size $(m_1,\dots,m_n)$ is chosen to be minimal, a magic rectangle of size $(m_1-4,m_2,\dots,m_n)$ exists. By Corollary 1, we then know a magic n-rectangle of size $(m_1,\dots,m_n)$ exists. Hence we can assume that $m_i\leqslant 6$ for all i. Since $(m_1,\dots,m_n)$ is a valid size, by Proposition 2 the only possibilities are, up to a permutation of the axes, $(2,6,\dots,6)$ and $(6,\dots,6)$. A magic n-rectangle of size $(2,6,\dots,6)$ exists by the hypotheses. We have already shown that a balanced centrally symmetric n-rectangle of size $(4,6,\dots,6)$ exists. So by Corollary 1, a magic n-rectangle of size $(6,\dots,6)$ exists. Hence a magic n-rectangle of size $(m_1,\dots,m_n)$ exists and the theorem has been proved.

In order to use Theorem 5 to prove Theorem 2, we need to show the existence of magic n-rectangles of size $(2,6,\dots,6)$. Inspection shows that

$$A=\begin{pmatrix} 1 & 11 & 3 & 9 & 8 & 7\\ 12 & 2 & 10 & 4 & 5 & 6\end{pmatrix}$$

is a magic 2-rectangle of size $(2,6)$ and that the array $B(x,M)=(b_{ijk})$ defined by

$$b_{1jk}=\begin{pmatrix} 1 & -4 & 13 & -14 & 36 & -32\\ 6 & -5 & 24 & -22 & 30 & -33\\ 12 & -11 & -21 & 23 & -31 & 28\\ -2 & 3 & -15 & 16 & 25 & -27\\ -8 & 7 & -19 & 17 & -26 & 29\\ -9 & 10 & 18 & -20 & -34 & 35\end{pmatrix}$$

and $b_{2jk}=-b_{1jk}$ is a balanced centrally symmetric 3-rectangle of size $(2,6,6)$. Hence magic rectangles of size $(2,6)$ and $(2,6,6)$ exist. We now show that a magic n-rectangle of size $(2,6,\dots,6)$ exists when n is odd. First we need to prove the proposition

Proposition 7. *If there exist balanced centrally symmetric rectangles of sizes* $(2,m_1,\dots,m_{n_1})$ *and* $(2,m_{n_1+1},\dots,m_{n_1+n_2})$, *then there exists a balanced centrally symmetric* n_1+n_2+1*-rectangle of size* $(2,m_1,\dots,m_{n_1+n_2})$.

Proof. Suppose $R_j=(r_i^{(j)})$ for $j=1,2$ are the two given balanced rectangles. Let $N_1=m_1\cdots m_{n_1}$ and $N_2=m_{n_1+1}\cdots m_{n_1+n_2}$. We can assume that the R_j have type 0. The entries of R_j are then $\pm1,\dots,\pm N_j$. We define a $(2,m_1,\dots,m_{n_1+n_2})$ array $T=(t_{aij})$ by

$$t_{aij}=|r_{1i}^{(1)}|+N_1(|r_{1j}^{(2)}|-1)]\mathrm{sgn}(r_{ai}^{(1)})\mathrm{sgn}(r_{1j}^{(2)}).$$

We note that since $r_{1i}^{(j)}+r_{2i}^{(j)}=0$, we have $|r_{1i}^{(j)}|=|r_{2i}^{(j)}|$. Now the entries of T are distinct. For if $t_{aij}=t_{bkl}$, then $|t_{aij}|=|t_{bkl}|$, which implies that $|r_{1i}^{(1)}|=|r_{1k}^{(1)}|$ and $|r_{1j}^{(2)}|=|r_{1l}^{(2)}|$. So $i=k$ and $j=l$. Further $\mathrm{sgn}(r_{ai}^{(1)})=\mathrm{sgn}(r_{bi}^{(1)})$, which shows $a=b$. Hence, the entries of T are the numbers $\pm1,\dots,\pm N_1N_2$. Now the row sums of T are all zero. The summation over the first index is zero since $r_{1i}^{(1)}=-r_{2i}^{(1)}$ and thus $t_{1ij}+t_{2ij}=0$. The summation over each of the i and j indices of T is 0 since $\mathrm{sgn}(r_i^{(j)})|r_i^{(j)}|=r_i^{(j)}$ and since the R_j are balanced rectangles, we have $\sum_{ik}r_{ai}^{(j)}=0,\sum_{ik}\mathrm{sgn}(r_i^{(j)})=0$. Hence, T is centrally symmetric n_1+n_2+1-rectangle. Since the R_j are balanced rectangles, the definition of the t_{aijk} shows that the rectangle T is also balanced.

Applying Proposition 7 repeatedly to the balanced centrally symmetric 2-rectangle (2,6,6), we have

Corollary 3. *Let* $n>1$ *be an odd number. Then there exists a balanced centrally symmetric* n*-rectangle of size* $(2,6.\dots,6)$.

Since balanced centrally symmetric rectangles correspond to magic n-rectangles, we have

Corollary 4. *For n an odd number, there exists a magic n-rectangle of size* $(2,6\ldots,6)$.

Hence a magic n-rectangle of size $(2,6,\ldots,6)$ exists when n is odd. To completely satisfy the hypotheses of Theorem 5, we also need to show that these magic n-rectangles exist when n is even. To construct these rectangles we will use:

Proposition 8. *If there exists a balanced centrally symmetric n-rectangle of size $(m_1,\ldots,m_{n-1},2)$, then there exists a magic $n+1$-rectangle of size $(6,m_1,\ldots,m_{n-1},2)$.*

Proof. Suppose $R=(r_i)$ is the given balanced rectangle. We can assume that R has type $-1/2$ and its entries consist of the numbers $\pm 1/2,\ldots,\pm(N-1/2)$, where $N=m_1\cdots m_{n-1}$. We define an array $S=(s_{aij})$ of size $(6,m_1,\ldots,m_{n-1},2)$ by

$$s_{ai}=\begin{cases} r_i+(a-1)N\operatorname{sgn}(r_i) & \text{for } a=1,2,3,\\ -r_i-(a-1)N\operatorname{sgn}(r_i) & \text{for } a=4,6,\\ \operatorname{sgn}(r_i)N-r_i+(a-1)N\operatorname{sgn}(r_i) & \text{for } a=5. \end{cases}$$

Since for a fixed a, the s_{ai} are distinct numbers and $(a-1)N<|s_{ai}|<aN$, the array S consists of the numbers $\pm 1/2,\ldots,\pm(6N-1/2)$. Further, since R is a balanced centrally symmetric rectangle, $\sum_{i_k} r_i=0$ and $\sum_{i_k}\operatorname{sgn}(r_i)=0$. Hence the row sums $\sum_{i_k} s_{ai}$ of S are zero. Finally, the row sum over a, $\sum_a s_{ai}$ is zero. Hence, S is a centrally symmetric rectangle of type $-1/2$. Corresponding to it is a magic $n+1$-rectangle of size $(6,m_1,\ \ldots,m_{n-1},2)$.

Combining Corollary 3 and Proposition 8, we have a magic n-rectangle of size $(2,6,\ldots,6)$ when n is even. Combined with Corollary 3, we have

Corollary 5. *For all n, there is a magic n-rectangle of size* $(2,6,\ldots,6)$.

Hence the hypotheses of Theorem 5 are established and Theorem 2 is proved.

4. Odd magic n-rectangles

We now consider the question of constructing odd magic n-dimensional rectangles. When $n=2$, it was shown in [4,6,7] that an odd magic rectangle of size (m_1,m_2) exists if $m_1,m_2>1$. We now use the two dimensional situation to construct magic rectangles of larger dimensions.

Lemma 2. *For odd $m_1, m_2 > 1$, there exists a $m_1 \times m_2$ rectangle whose entries are the integers $[1, m_1]$, each appearing m_2 times, such that the entries of each column are distinct and each row has the same sum.*

Proof. We then define a $m_1 \times m_2$ rectangle $R = (r_{ij})$ by $1 \leqslant r_{ij} \leqslant m_1$ and the mod m_1 congruence

$$r_{ij} \equiv \begin{cases} 1 - 2i & \text{if } j = 1, \\ i + (m_1 + 1)/2 & \text{if } j = 2, \\ i & \text{if } j \text{ odd, } j > 1, \\ 1 - i & \text{if } j \text{ even, } j > 2. \end{cases}$$

Each column of R contains the first m_1 positive integers and the row sums $\sum_j r_{ij}$ are easily checked to be constant.

Proposition 9. *Let $m_1, m_2 > 1$ be odd integers. Then a magic 3-rectangle of size (m_1, m_1, m_2) exists.*

Proof. Let $R = (r_{ij})$ be a magic 2-rectangle of size (m_1, m_2). Let $T = (t_{ij})$ be the $m_1 \times m_2$ rectangle constructed in Lemma 2. We define an array $S = (s_{ijk})$ of size (m_1, m_1, m_2) by $s_{ijk} = r_{i+j,k} + m_1 m_2 (t_{i-j,k} - 1)$, where the indices $i+j, i-j$ are viewed as numbers mod m_1. Now $1 \leqslant s_{ijk} \leqslant m_1^2 m_2$ and $s_{ijk} = s_{i'j'k'}$ implies that $r_{i+j,k} = r_{i'+j',k'}$. Hence $k = k'$ and $i + j \equiv i' + j'$ mod m_1. Also, $t_{i-j,k} = t_{i'-j',k}$ and since T has distinct entries in each column, $i - j \equiv i' - j'$ mod m_1. Hence $i = i'$ and $j = j'$ and the entries of T are distinct. Since both R and T have constant row and column sums, S does as well and S is a magic 3-rectangle.

As an example, Proposition 9 shows that a magic 3-rectangle of size $(3, 3, c)$ exists. We now show that there is a scalar multiplication on odd 3-rectangles. This generalizes Theorem 2 of [3] and proves that 3-rectangles of size $(3a, 3b, c)$ always exist.

Proposition 10. *Let α be a positive odd integer. If a magic 3-rectangle of size (m_1, m_2, m_3) exists, then a magic 3-rectangle of size $(\alpha m_1, m_2, m_3)$ exists.*

Proof. Let $R = (r_i)$ be a magic 3-rectangle of size (m_1, m_2, m_3). We will construct α rectangles $T_a = (t_{ai})$ of size (m_1, m_2, m_3) with the following properties. First, the row sums of the T_a are constant and are independent of a. Second, for fixed i, j, k, the set $\{t_{aijk}\}$ contains the first α positive integers. As a consequence, the union of the entries of the T_a contains each of the numbers $1, \ldots, \alpha$ exactly $m_1 m_2 m_3$ times. We define a 3-rectangle $S = (s_{ijk})$ of size $(\alpha m_1, m_2, m_3)$ by $s_{ijk} = r_{i_0 jk} + m_1 m_2 m_3 (t_{a i_0 jk} - 1)$, where $i = (a-1) m_1 + i_0$, and $1 \leqslant i_0 \leqslant m_1$. Together, the two properties on the T_a show that S is a magic 3-rectangle of size $(\alpha m_1, m_2, m_3)$. We now show that the desired rectangles

T_a can be constructed. Define the function c_{an} by $1 \leqslant c_{an} \leqslant \alpha$ and

$$c_{an} \equiv \begin{cases} 1+2a \bmod \alpha, & n \equiv 1 \bmod 3, \\ \dfrac{\alpha+1}{2} - a \bmod \alpha, & n \equiv 2 \bmod 3, \\ \alpha - a \bmod \alpha, & n \equiv 3 \bmod 3. \end{cases}$$

We define the matrices $T_a = (t_{aijk})$ for $k \leqslant 3$ by

$$t_{aijk} = \begin{cases} c_{a,i+j+k}, & i,j \leqslant 3, \\ c_{a,j+k}, & i \text{ even}, \ j \leqslant 3 < i, \\ \alpha + 1 - c_{a,j+k}, & i \text{ odd}, \ j \leqslant 3 < i, \\ c_{a,i+k}, & j \text{ even}, \ i \leqslant 3 < j \\ \alpha + 1 - c_{a,i+k}, & j \text{ odd}, \ i \leqslant 3 < j, \\ c_{a,k}, & i+j \text{ even}, \ 3 < i,j, \\ \alpha + 1 - c_{a,k}, & i+j \text{ odd}, \ 3 < i,j, \end{cases}$$

and for $k > 3$ by

$$t_{aijk} = \begin{cases} t_{aij1}, & k \text{ even}, \\ \alpha + 1 - t_{aij1}, & \text{otherwise}. \end{cases}$$

Explicit calculation shows that for each a, the row sums of T_a are constant and the first condition on the T_a is satisfied. Moreover, for fixed n, the function c_{an} represents each of the integers $[1, \alpha]$, and so for fixed i, j, and k, the function t_{aijk} represents each of the integers $[1, \alpha]$. Thus the matrices T_a satisfy the second condition and the proposition is proven.

Combining Propositions 9 and 10 immediately gives Theorem 3. It also has as a more concrete corollary:

Corollary 6. *Let p be an odd prime and let $a, b, c > 0$ be odd integers with $c > 1$. Then a magic 3-rectangle of size (ap, bp, c) exists.*

Proposition 10 also reduces the problem of determining which odd magic 3-rectangles exist to the problem of determining which magic 3-rectangles with prime dimensions exist.

Corollary 7. *Suppose that for all prime numbers p, q, and r, a magic 3-rectangle of size (p, q, r) exists. Then for all odd integers $m_1, m_2, m_3 > 1$, a magic 3-rectangle of size (m_1, m_2, m_3) exists.*

The simplest case in which to check whether the hypotheses of Corollary 7 hold is that of a magic 3-rectangle of size $(3, 5, 7)$. The computations involved are quite large however, and the existence of a magic 3-rectangle of size $(3,5,7)$ remains an open question.

5. For further reading

The following reference is also of interest to the reader: [5].

Acknowledgements

The author would like to express his thanks to Dan Shapiro for his help and encouragement and for suggesting this problem, and to Allan Adler and the referees.

References

[1] A. Adler, Magic cubes and the 3-adic zeta function, Math. Intelligencer 14 (3) (1992) 14–23.
[2] A. Adler, L. Washington, p-adic L-functions and higher-dimensional magic cubes, J. Number Theory 52 (2) (1995) 179–197.
[3] T. Bier, G. Rogers, Balanced magic rectangles, European J. Combin. 14 (1993) 285–299.
[4] T. Bier, A. Kleinschmidt, Centrally symmetric and magic rectangles, Discrete Math. 176 (1997) 29–42.
[5] A. Evans, Magic rectangles and modular magic rectangles, J. Statist. Plann. Inference 51 (2) (1996) 171–180.
[6] T. Hagedorn, Magic rectangles revisited, Discrete Math. 207 (this Vol.) (1999) 65–72.
[7] T. Harmuth, Über magische Quadrate und ähniche Zahlenfiguren, Arch. Math. Phys. 66 (1881) 286–313.
[8] T. Harmuth, Über magische Rechtecke mit ungeraden Seitenzahlen, Arch. Math. Phys. 66 (1881) 413–447.
[9] R. Sun, Existence of magic rectangles, Nei Mongol Daxue Xuebao Ziran Kexue 21 (1) (1990) 10–16.
[10] R. Sun, The existence of higher-dimensional magic cubes, Neimenggu Daxue Xuebao 19 (2) (1988) 213–223.